Perturbative Improvement for Lattice QCD: An Update*

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Abstract

Recent developments in the Symanzik improvement program for lattice QCD are reviewed.

1 Introduction

In this lecture I review recent trends in the effort to develop highly improved discretizations of QCD for use on coarse lattices. I focus here on the perturbative Symanzik approach to this problem; other approaches are discussed by other lecturers at this meeting. The ultimate goal of all this activity is too allow us to simulate with larger lattice spacings, at greatly reduced cost. The savings that result from larger lattice spacings can be used to extend the reach of lattice QCD to far more difficult problems, well suited to our largest supercomputers. At the same time, there is a wealth of simpler simulation problems that are now accessible to anyone with a modern workstation or personal computer.

It is somewhat artificial to isolate a single approach like Symanzik improvement. Symanzik improvement is but one of a large set of tools — Symanzik and on-shell improvement, perturbation theory and tadpole improvement, blocked fields and perfect actions, nonperturbative tuning and MCRG, and so on — that we can use to design improved discretizations. These tools are closely related, and not mutually exclusive. Solving QCD is much too difficult for us to ignore any particular set of tools; we need every trick available.

I begin, in Section 2 with a review of perturbative improvement and tadpole improvement. I discuss variations on the canonical approach, including nonperturbative tuning, and I discuss what we now know about the size of finite-a errors. Weak-coupling perturbation theory plays a central role in all practical

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approaches to improved discretization. I discuss the status of lattice perturbation theory, as well as how to implement and test it, in Section 3. In Section 4 I discuss specific, recent developments concerning various actions. Finally I summarize in Section 5.

2 Perturbative and Tadpole Improvement

2.1 The Method

The procedure for perturbatively improving a lattice QCD operator or action involves three steps [1]:

1. Classical Improvement: The classical (tree-level) lattice action or operator is corrected through the desired order in the lattice spacing a. For example, the lattice gluon action

$$\begin{aligned} S &= -\beta \sum_{x,\mu>\nu} \{c_1 P_{\mu,\nu}(x) \\ &\quad + c_2 \left(R_{\mu\mu,\nu}(x) + R_{\nu\nu,\mu}(x)\right)\} \end{aligned} \tag{1}$$

   equals

$$\int d^4x \, \tfrac{1}{2} \mathrm{Tr} F_{\mu\nu}^2(x) + \mathcal{O}(a^4) \tag{2}$$

   for classical fields when

$$\beta = 6/g^2, \quad c_1 = 5/3, \quad c_2 = -1/12. \tag{3}$$

   Here $P_{\mu,\nu}$ is one third the real part of the trace of the 1×1 Wilson loop in the $\mu\nu$ plane, while $R_{\mu\mu,\nu}$ is the same for a 2×1 loop with two steps in the μ direction and one in the ν direction. We derive this result by substituting the definition of the lattice link, in terms of the continuum vector potential,

$$U_\mu(x) = \mathrm{P} \exp\left[-i \int_0^a d\xi \, g A_\mu(x + \xi\hat{\mu})\right], \tag{4}$$

   into the lattice action, and expanding in powers of the lattice spacing a.

2. Tadpole Improvement: In the quantum theory, the couplings are renormalized by quantum fluctuations, and new operators are required. The dominant renormalizations are due to tadpole diagrams caused by the (nonlinear) gluon link operator U_μ. These lattice artifacts are largely cancelled by tadpole improvement: every link in the classical lattice theory is divided by the mean link [2, 1]:

$$U_\mu(x) \to \frac{U_\mu(x)}{u_0}. \tag{5}$$

The mean link u_0 is a number between 0 and 1; it is computed self-consistently, using a Monte Carlo simulation, from the mean value of $\frac{1}{3}\mathrm{Tr}U_\mu$. This last quantity is gauge dependent and can be made arbitrarily small by changing the gauge. Thus we choose the gauge that maximizes $\langle \mathrm{Tr}U_\mu \rangle$, which is Landau gauge:

$$u_0 \equiv \langle 0|\tfrac{1}{3}\mathrm{Tr}U_\mu|0\rangle_{\mathrm{LG}}. \tag{6}$$

This gauge minimizes the tadpoles in u_0, pushing it as close to 1 as possible; any tadpole contribution that remains cannot be a gauge artifact. Our gluon action, Eq. (1), is tadpole improved by setting

$$c_1 = \frac{5}{3\,u_0^4}, \qquad c_2 = -\frac{1}{12\,u_0^6}. \tag{7}$$

The tadpole correction here is substantial at lattice spacings of .25–.4 fm since u_0 is .7–.8 and is raised to large powers.

At small lattice spacings u_0 is well approximated by the fourth root u_p of the plaquette operator, which is easier to compute than the mean link in Landau gauge:

$$u_0 \approx u_p \equiv \langle 0|P_{\mu,\nu}|0\rangle^{1/4}. \tag{8}$$

3. Radiative Corrections: A tadpole-improved classical action or operator is an efficient starting point for a systematic weak-coupling expansion. Each coupling constant has an expansion in powers of the QCD coupling constant α_V:

$$c_i = \frac{1}{u_0^{n_i}}\left(c_{i0} + c_{i1}\,\alpha_V(q_i^*) + \cdots\right) \tag{9}$$

The scale q_i^* is typically between 0.5/a and 2/a, and is easily computed using one-loop perturbation theory [2]. The value of the coupling constant is also easily measured in the simulation [2]. New operators, beyond those necessary in the classical analysis, are usually needed when radiative corrections are included. The perturbative expansions of the coupling constants are computed by matching: low-energy physical quantities, scattering amplitudes for example, are computed perturbatively in both the lattice theory and the continuum, and lattice couplings are tuned so that the two calculations agree order-by-order in the coupling (to a given order in the lattice spacing). The perturbation expansion is usually much more convergent with the tadpole factor, $1/u_0^{n_i}$, removed as above. Indeed, for many applications, only the tree-level term in the expansion is necessary after tadpole improvement.

The procedure just outlined is a double expansion in powers of the lattice spacing and of the QCD coupling constant. Carried to all orders it becomes infinitely complicated; in practice, however, only a few low orders of correction

are needed to reduce errors to a few percent. Recently we have started to obtain high-quality data on the size of finite-a errors. An example is the SCRI analysis of errors in the hadron spectrum for Wilson and SW quarks [3]. By going to large a, they made the errors large and unambiguous; they tested the algorithms "to failure." They find that the relative errors are roughly $(a/a_0)^n$ with $n=1$ for Wilson quarks and $n=2$ for SW quarks, and $a_0 \approx$1–2 fm. The powers are as expected, and so is the scale a_0 for the errors [4].

Such information tells us how much improvement we need at a given lattice spacing. At $a = .4$ fm, for example, $a/a_0 \approx .4$ and $\alpha_V \approx .3$. Thus, if we want errors smaller than 5–10%, we should include in our quark action tree-level corrections through order a^3 and radiative corrections through order $a\,\alpha_V$ (that is, radiative corrections for only the order a term, and then only through one-loop). At $a = .1$ fm, on the other hand, only the order a and, perhaps, the order $a\,\alpha_V$ corrections are necessary for 1% accuracy (in addition to 4^6 times as much computing). Note that as a increases higher powers of a become more important faster than higher powers of α_V. These arguments are approximate, but they provide a starting point for a realistic assessment of what is needed.

Radiative corrections (beyond tadpole improvement) are unimportant for most terms in quark and gluon actions. This is because the radiative corrections are in terms already suppressed by some power of the lattice spacing; radiative corrections for terms of order a^0 are generally absorbed into masses or other parameters that are tuned numerically. The situation is different, however, for lattice operators like currents. Typically the desired continuum current is obtained from a lattice current using

$$J_{\rm cont} = Z(\alpha_V)\; J_{\rm lat} + \gamma_1(\alpha_V)\, a\, \delta J_1 + \mathcal{O}(a^2). \tag{10}$$

While radiative corrections may be unimportant in the $\mathcal{O}(a, a^2 \ldots)$ corrections, the Z-factor appears in order a^0 and therefore must usually be determined through one or two orders in α_V to achieve useful accuracy. The problem is even more accute for operators that mix through radiative corrections with operators of lower dimension, because of $1/a$ divergences; this can be alleviated to some extent by working with larger a's, where $1/a$ is then smaller.

An alternative to tadpole improvement worth investigating is to replace the link operators in quark actions and currents by smeared links: for example,

$$U_\mu \to \tilde{U}_\mu \equiv \left[1 + \frac{\epsilon\, a^2 \Delta^{(2)}}{n}\right]^n U_\mu \tag{11}$$

where $\Delta^{(2)}$ is the discretized gauge-covariant laplacian, n is a small integer, and ϵ a smearing parameter. This introduces a factor like $\exp(-\epsilon\, a^2 q^2)$ into the gluon propagator which reduces the effective ultraviolet cutoff for gluons coupled to the quarks. Since tadpoles are quadratically divergent, even a modest reduction in this cutoff greatly reduces their contribution and makes tadpole improvement unnecessary. Substituting $\tilde{U}_\mu$ for U_μ in a lattice operator introduces a new

a^2 error; this can be removed by including an additional factor of $(1 - \epsilon\, a^2\Delta^{(2)})$ in the definition of $\tilde{U}_\mu$.

2.2 Nonperturbative Tuning

A variation on the standard procedure is to tune some of the couplings nonperturbatively, using simulations. There are two standard approaches. One is to tune a coupling constant to restore a continuum symmetry. In many cases a leading correction term breaks a continuum symmetry, like Lorentz invariance, and is the only term of its order (in a) to do so. Then the coefficient of that term can be determined by adjusting it until the corresponding symmetry is restored in the simulation results. For example, there is only one independent dimension-six operator that breaks rotation invariance and contributes to the $\mathcal{O}(a^2)$ corrections in a gluon action. Thus we can adjust coupling c_2 in our lattice gluon action, Eq. (1), until, for example, the simulation gives a static quark potential with $V(2r, 2r, r) = V(3r, 0, 0)$ for some r. The tuned coupling then automatically includes perturbative corrections to all orders in α_V, as well as any nonperturbative contributions.[1] The tuned coupling is not exact, however, since the quantities used to tune it have finite-a errors caused by the action. In our example, the gluon action is accurate only through order a^2. Thus $V(2r, 2r, r) - V(3r, 0, 0)$ has order a^4 errors; tuning it to zero induces order a^4 errors in c_2. Obviously, such truncation errors are smaller for more highly corrected lattice theories. Also some care is necessary in choosing the quantitiy to be tuned; one wants a quantity for which finite-a errors are minimal or small.

Other examples of couplings that can be tuned using symmetry restoration include: the clover coefficient in the Sheikholeslami-Wohlert (SW) quark action, using chiral symmetry [5]; the coefficient of the $\overline{\psi}\Delta^{(3)}\cdot\gamma\psi$ correction in the D234 actions, using Lorentz invariance (see below); and κ_t/κ_s in the Fermilab quark action for massive quarks, using Lorentz invariance.

The second approach to nonperturbative tuning is to adjust couplings so that physics remains the same at different lattice spacings. Thus, for example, one might adjust the clover coefficient in the SW action so that the rho mass scales correctly from lattice spacing to lattice spacing. This is harder than tuning for a symmetry since simulations at several lattice spacings are required, but it is not impossibly difficult if only one or two couplings are needed. One loses one "prediction" from the simulation for each coupling that is tuned this way. Again the couplings that result include all radiative corrections and any nonperturbative effects; and, again, the couplings have finite-a errors caused by the action.

[1] Any short-distance quantity could have nonperturbative contributions. Usually these are small compared with the perturbative contributions, which is why perturbative QCD works so well. Note that lattice coupling constants have renormalon contributions because of finite-a errors in the lattice action; but these contributions are automatically suppressed to the same level as other finite-a errors caused by the action, and so can be ignored.

Nonperturbative tuning is particularly useful for leading-order corrections, like the order-a term in quark actions, where there are only one or two couplings to tune. Where it can be used, it is also useful for currents since otherwise one-loop and sometimes two-loop radiative corrections must be computed. For finite-a corrections beyond leading order, however, tadpole improvement and perturbation theory are likely to be the most feasible approach. Fortunately, tree-level perturbation theory is probably adequate for most correction terms in order a^2 and higher.

3 Perturbation Theory

3.1 Does It Work?

Weak-coupling perturbation theory plays an essential role in all current approaches to improved lattice QCD actions and operators. It tells us that improving the classical theory is a good starting point for the quantum improvement program. It tells us which corrections are most important; as a result we need only a small number of correction terms. In the perturbative improvement schemes I discuss here, perturbation theory is the tool for calculating the coupling constants. But can we trust perturbation theory?

The short answer is yes. Wherever careful tests have been made, using perturbative expansions through second or third order, for quantities that are easy to measure accurately in simulations, perturbation theory has been successful. For example, in [6] four values for the QCD coupling constant α_P are extracted using third-order expansions for four different Wilson loops — W_{11}, W_{12}, W_{13}, and W_{22} — at lattice spacing $a \approx 0.1\,\mathrm{fm}$ ($\beta = 6$). These agree with each other to within the estimated truncation errors, $O(\alpha_P^4) \approx 1\%$; and they agree to the same precision with the value obtained from the (very different) Schrödinger Functional approach [8]. This is a high-precision test of QCD perturbation theory, much more stringent than any test based on continuum quantities. Another three-loop test is coupling constant evolution: As illustrated in Fig 1, the running of the QCD coupling constant is well described by the perturbative beta function at least down to scales of order 5 GeV; less accurate tests show this is true even down to scales of order 1 GeV [2].

Usually when we use perturbation theory, we truncate the expansion at some finite order in α_V. We should always estimate the truncation error that results. Sometimes an apparent disagreement between perturbation theory and simulation results is negligible when compared with the truncation error, and is therefore of little significance. For example, the critical κ for SW quarks in tadpole-improved perturbation theory is known only through first-order [5, 7]:

$$\kappa_c = \frac{1}{8u_p}\left[1 - 0.20\,\alpha_V(4.4/a) + \mathcal{O}(\alpha_V^2)\right] \tag{12}$$

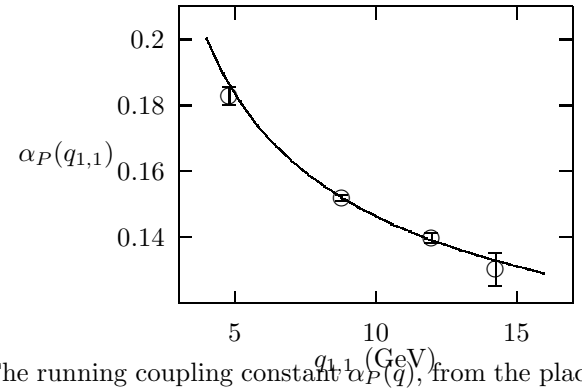

Figure 1: The running coupling constant $\alpha_P(q)$, from the plaquette at different lattice spacings, plotted versus the typical momentum, $q_{1,1} = 3.4/a$, in the plaquette; lattice spacings are from the $1S$–$1P$ upsilon splittin in NRQCD. The solid line is predicted by the three-loop beta function.

where u_p is the plaquette mean link. Thus perturbation theory should only be correct up to relative errors of order $\pm 1\alpha_V^2$. In Figure 2, we compare the first-order result, including error bars for higher-order terms, with nonperturbative results obtained from simulations [5]. The nonperturbative results are more accurate, but perturbation theory agrees well — that is, it agrees to within truncation errors.

It is equally important, when using nonperturbative tuning, to estimate the finite-a errors due to truncation of the lattice action. The clover coefficient $c_{\rm sw}$ in the SW action can be tuned nonperturbatively using chiral symmetry [5]. The tuned $c_{\rm sw}$, however, has truncation errors of relative order a/a_0 for some a_0 around 1–2 fm, at least for larger lattice spacings (at smaller lattice spacings one can compute the order a term, using perturbation theory, and remove it). The tuned coupling can be compared with the perturbative result [5],

$$c_{\rm sw} = \frac{1}{u_p^3}\left[1 + 0.20\,\alpha_V(q^*_{\rm sw}) + \mathcal{O}(\alpha_V^2)\right], \tag{13}$$

which also has truncation errors, of relative order $\pm 1\alpha_V^2$. Scale $q^*_{\rm sw}$ is currently unknown, but is likely around $.5/a$–$2/a$. In Fig. 3 we compare the nonperturbative and perturbative results, indicating the uncertainties in each, for a range of β's. (For this we guess $a_0 = 2\,{\rm fm}$ and $q^*_{\rm sw} = 1/a$). The perturbative and nonperturbative $c_{\rm sw}$'s agree well, and are roughly equal in accuracy; each reinforces our confidence in the other. Note that, since $c_{\rm sw}$ is the coefficient of an $\mathcal{O}(a)$ correction, the different $c_{\rm sw}$'s at $\beta = 6$ give identical hadronic physics to within 1–2%.

We also include results in the last figure from first-order perturbation theory,

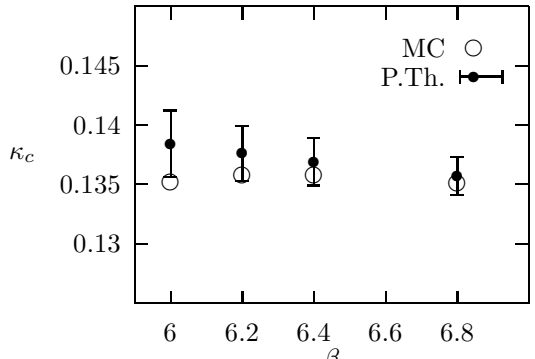

Figure 2: Critical κ for SW quarks from Monte Carlo simulations and from first-order perturbation theory. Statistical errors are negligible for the Monte Carlo results; relative errors shown for the perturbation theory are $\pm 1\alpha_V^2$.

but with the Landau-gauge mean link u_0 instead of u_p:

$$c_{\rm sw} = \frac{1}{u_0^3}\left[1 + 0.46\,\alpha_V + \mathcal{O}(\alpha_V^2)\right]. \tag{14}$$

While u_0 and u_p are almost identical through order α_V, they differ in higher order. Thus differences between the two options for tadpole improvement indicate the potential size of order α_V^2 corrections; these differences are 5–10 times smaller if α_V^2 corrections are included in the two $c_{\rm sw}$ expansions.

Perturbation theory triumphs in these and most other examples; it is usually neither more nor less accurate than expected (from perturbation theory). Nevertheless there will be situations where perturbation theory fails to converge. In QED, for example, the decay rate for orthopositronium is proportional to $\alpha^6\,(1 - 10.3\alpha/\pi)$ [9]. This expansion would be useless if, in QED, $\alpha = 0.2$ or 0.3. QCD perturbation theory will fail at distances smaller than 0.2 fm, but not often and not by much; one should worry about one's simulation if it disagrees with perturbation theory at such distances.

3.2 Keeping Up

Improved lattice actions and operators can be quite complicated, and lately they have been changing rapidly. Unfortunately perturbative calculations of coupling constants and renormalization constants must be redone if any of the actions or operators involved is changed. Also the vertices and propagators can be quite complicated in a highly improved theory, making even one-loop calculations a chore. The perturbative work on NRQCD, which has a relatively complicated action, demonstrates that such calculations are possible [10]. We need to systematize such calculations so that one-loop calculations, at least,

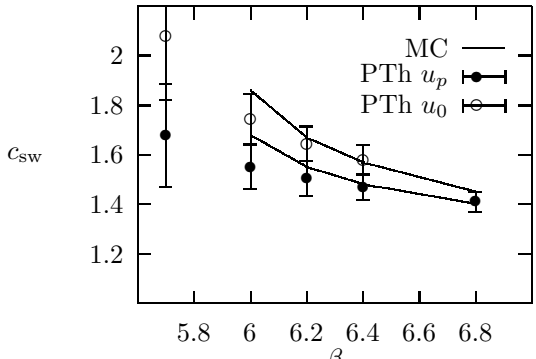

Figure 3: Clover coefficient $c_{\rm sw}$ for SW quarks from Monte Carlo simulations, and from first-order perturbation theory using the plaquette mean link u_p and the Landau-gauge mean link u_0. Relative errors shown for the perturbation theory are $\pm 1\alpha_V^2$. The lines indicate the possible range of the nonperturbative $c_{\rm sw}$ (due to finite-a errors) around the parameterization given in [5].

can be completed in a short time for any new quark or gluon action. Otherwise we will find ourselves tied to older, probably much less efficient actions and operators by the lack of perturbative calculations for the new ones. Substantial progress in this direction is being made now, but more effort is needed.

One approach that deals with this problem is to use Monte Carlo simulations to do the lattice perturbation theory [11]. On-shell quark form factors, renormalization constants, masses, etc. can be computed using simulations at very small lattice spacings, for example $a \approx 0.002\,\text{fm}$ at $\beta = 9$, and matched directly with continuum calculations. Since confinement is irrelevant at such short distances, coupling constants determined in such a matching can be fit by polynomials in α_V (which is also measured in the simulation). In this way we obtain perturbative expansions for the couplings that can be used in simulations at more normal lattice spacings.

4 Recent Developments

4.1 Landau-Gauge Mean Link

It is standard practice to approximate the Landau-gauge mean link u_0 by the plaquette mean link u_p. When the lattice spacing is small, either of these is a reasonable choice. The two definitions differ, however, by almost 10% when $a = 0.4\,\text{fm}$ (probably due mostly to perturbative differences in $\mathcal{O}(\alpha_V^2)$). While coupling constants obtained from the two definitions agree when perturbative corrections are included to all orders, tree-level or one-loop results can be quite

different when a is large. Three pieces of evidence suggest that it is better to stay with the Landau-gauge mean link u_0. First the tree-level, tadpole-improved gluon action in Eq. (1) shows 2–3 times less violation of rotation invariance (in the static potential) when u_0 is used rather than u_p [12]. Second, scaling violations in the charmonium groundstate hyperfine splitting are 3 times smaller when the tree-level $\sigma \cdot B$ interaction is tadpole-improved with u_0 rather than u_p [13]. Finally, the nonperturbative determination of the clover coefficient $c_{\rm sw}$ for SW quarks agrees more closely with tadpole-improved one-loop perturbation theory when u_0 is used, as is evident in Fig. 3. The same has also been observed at larger lattice spacings, with improved gluon actions, and also in improved quark actions like the D234c action discussed below [14]. These examples suggest that u_0 might be the better choice generally, at least for large lattice spacings, if only tree-level or first-order corrections are included; there is little difference if corrections through second order are included.

4.2 Anisotropic Lattices

It is widely felt that coarse lattices are noisier than fine lattices. This is correct, especially for correlation functions where signal/noise vanishes expontially quickly with increasing separation t. It is true despite the fact that the ratio of signal to noise is approximately independent of lattice spacing when t is large; noise overcomes signal at some fixed physical distance, independent of the lattice spacing. The problem on coarse lattices is that this fixed distance might be only one or two lattice spacings long, since the lattice spacing is large. Consequently, without large Monte Carlo data sets, the statistically usable part of a mass plateau, for example, may be too short to allow identification or verification of the plateau. A lattice with small lattice spacing allows us to identify the plateau at smaller distances, with less noise, but at increased cost from the small lattice spacing. Since signal/noise decreases exponentially with increasing t, while simulation cost grows only as a (large) power of lattice spacing, one might decide that smaller lattice spacings are ultimately more efficient. Much of the advantage of the larger lattice spacing is regained by careful design of sources and sinks, but a far more efficient approach is to reduce the lattice spacing a_t in the time direction while retaining a large spacing a_s in spatial directions. Then more values of a correlation function can be measured at small t's, where relatively few Monte Carlo measurements are needed to suppress the noise. The small increase in simulation cost that results from a smaller a_t is more than offset by the exponential improvement in signal-to-noise at the smaller t's.

Anisotropic actions are easy to design. The standard improved gluon action becomes [15, 1]

$$S_n = -\beta \sum_{x,\, s>s'} \frac{a_t}{a_s}\, \eta \left\{ \frac{5}{3} \frac{P_{s,s'}}{u_s^4} \right. \tag{15}$$

$$-\frac{1}{12}\frac{R_{ss,s'}}{u_s^6}-\frac{1}{12}\frac{R_{s's',s}}{u_s^6}\Bigg\} \tag{16}$$
$$-\beta\sum_{x,\,s}\frac{a_s}{a_t}\,\eta^{-1}\,\left\{\frac{4}{3}\frac{P_{s,t}}{u_s^2u_t^2}-\frac{1}{12}\frac{R_{ss,t}}{u_s^4u_t^2}\right\},$$

where u_s and u_t are the (Landau-gauge) spatial and temporal mean links, respectively, and η accounts for renormalization of the anisotropy by quantum effects. Since $a_t \ll a_s$, we have not bothered to correct for order-a_t^2 errors in this action; thus the action extends at most one step in the time direction and has no gluon ghosts. An a^2-improved version of Landau gauge is obtained by maximizing

$$\sum_{x\,\mu}\frac{1}{u_\mu a_\mu^2}\,\mathrm{ReTr}\left\{U_\mu(x)-\frac{1}{16u_\mu}\,U_\mu(x)\,U_\mu(x+\hat{\mu})\right\}. \tag{17}$$

The anisotropic gluon action seems to have six parameters, but these appear in only three independent combinations,

$$\frac{\beta}{u_s^3u_t},\qquad\left(\frac{\eta a_t}{a_s}\right)\frac{u_t}{u_s},\qquad u_s, \tag{18}$$

which control the spatial lattice spacing, the anisotropy, and the size of the order a_s^2 corrections, respectively. These can be tuned numerically by choosing β and $\eta a_t/a_s$, then determining u_t and u_s self-consistently, and finally measuring η. Alternatively one might compute the last three parameters using perturbation theory. In [15] simulations with very small lattice spacing are combined with first-order perturbation theory to obtain (prelimary) perturbative expressions for the case $\eta a_t/a_s = 0.5$:

$$u_s = 1-0.517/\beta+.17(5)/\beta^2 \tag{19}$$
$$u_t = 1-0.105/\beta-.02(1)/\beta^2 \tag{20}$$
$$\eta = 1-0.114(7)/\beta. \tag{21}$$

These expressions give good results even at $a=.4\,\mathrm{fm}$, as shown in Table 1.

In Fig. 4 we show the static potential computed on an anisotropic lattice from Wilson loops involving temporal and spatial links, and from loops involving just spatial links. Shifted to account for renormalization, the two potentials agree well for all r, as they should.

Anisotropic lattices with $a_t \ll a_s$ are useful for any simulation involving high-energy states, including glueballs, excited hadronic states, and high-momentum final states in form factors. The dominant lattice errors in relativistic heavy-quark physics are powers of a_tM_Q, where M_Q is the mass of the heavy quark; these are greatly reduced on highly anisotropic lattices, which, therefore, provide an alternative to the NRQCD and Fermilab approaches [14]. A variety of quark and gluon actions designed for anisotropic lattices are currently under development.

		$\beta = 1.8$	$\beta = 1.34$
u_s	P.Th.	.77(2)	.71(3)
	M.C.	.75	.69
u_t	P.Th.	.94(1)	.91(1)
	M.C.	.93	.91
η	P.Th.	.94(1)	.91(2)
	M.C.	.94(1)	.90(1)

Table 1: Monte Carlo and perturbation theory values for parameters in the anisotropic gluon action with $\eta a_t/a_s = 0.5$. Results are given for two lattice spacings (approximately 0.25 fm and .4 fm), and are preliminary.

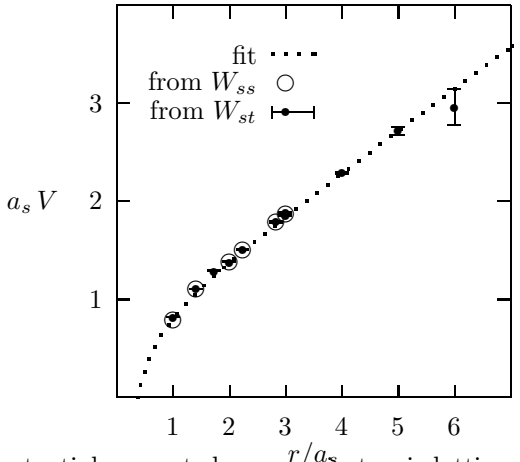

Figure 4: Static potential computed on an anisotropic lattice with $\eta\, a_t/a_s = 0.5$ and $\beta = 1.8$. Potentials obtained from spatial-temporal loops and from purely spatial loops are shown. Results are preliminary.

4.3 Improved Gluons

The plaquette-plus-rectangle gluon actions, Eqs. (1) and (17), have been used by several groups in recent years for simulations of the static potential, QCD thermodynamics, charmonium, the light-hadron spectrum, and so on. In every application area, except perhaps the light-hadron spectrum, the improved action has led to significantly improved results.

One of the most striking recent applications is in [16], a high-statistics study of several different glueball masses at a wide range of lattice spacings. What is striking is that this thorough study was carried out entirely on desktop workstations, an achievement possible only through the use of coarse, highly anisotropic (5:1) lattices. All the glueball masses extrapolate to the values determined earlier with more traditional methods; and all but one show relatively little dependence on the lattice spacing, less than 10%, out to lattice spacings of order 0.4 fm. Only the 0^{++} glueball shows strong a dependence; its mass has a 20–25% dip around 0.3 fm. These are the largest measured finite-a errors in gluonic lattice QCD, and therefore the 0^{++} glueball is of particular interest to designers of improved gluon actions.

A dip in the 0^{++} mass is also seen in SU(2) QCD [17], although it is smaller, only about 14%. The authors of the SU(2) analysis noticed that the depth of the dip is reduced almost by half when the Landau-gauge mean link u_0 is used, in place of the plaquette mean link u_p, in the tadpole-improved tree-level gluon action. This result is consistent with the observation that the static-quark potential is more rotationally invariant with u_0; both argue that the correct order a^2 correction in the action is stronger than that obtained using u_p. The SU(3) study used u_p; it should be redone with u_0 instead. Better still, we should calculate the full $\mathcal{O}(a^2\alpha_V)$ corrections to the anisotropic gluon action; work on this has begun. A small glueball radius would explain the strong a dependence of the 0^{++} mass; another explanation is that the dip in the mass is caused by a phase transition nearby in the fundamental-adjoint coupling constant plane of the gluon action.

4.4 Improved Wilson Quarks

Many of the talks at this meeting showed results for the SW quark action, which is an order a improved version of the original Wilson lattice action. Results are clearly superior when the SW action is used instead of the Wilson action, provided the clover coefficient is either tadpole improved or tuned nonperturbatively; and generally SW and Wilson simulations extrapolate to the same answer. Based on this work, the SW action should be accurate to within a few percent for light-quark physics out to lattice spacings of order 0.15–0.2 fm.

Larger lattice spacings require a more accurate action. A simple construction that yields such an action begins with the continuum lagrangian:

$$\mathcal{L} = \overline{\psi}_c M_c \psi_c, \qquad M_c \equiv D \cdot \gamma + m_c. \tag{22}$$

Still working in the continuum, we introduce a field transformation,

$$\begin{aligned} \psi_c &= \Omega\psi & (23)\\ \Omega^2 &= 1 - \tfrac{1}{2}ra(D\cdot\gamma - m_c) & (24) \end{aligned}$$

to obtain a new lagrangian $\overline{\psi}M\psi$ where

$$\begin{aligned} M &= \Omega M_c \Omega \\ &= D\cdot\gamma + m_c\left(1+\tfrac{1}{2}ram_c\right) & (25)\\ &\quad -\tfrac{1}{2}ra\left(D^2 + \tfrac{1}{2}g\sigma\cdot F\right). & (26) \end{aligned}$$

The third term in M, introduced by the field transformation, breaks the doubling symmetry when we discretize the action. Allowing errors only of order a^4 and higher, we obtain the D234c discretization of M [14]:

$$\begin{aligned} M_{\rm D234} &\equiv \Delta\cdot\gamma - c_3\,\frac{a^2}{6}\Delta^{(3)}\cdot\gamma + m & (27)\\ &- \frac{ra}{2}\left(\Delta^{(2)} + c_F\frac{g\sigma\cdot F}{2} - c_4\frac{a^2}{12}\Delta^{(4)}\right) \end{aligned}$$

where $m \equiv m_c(1+\frac{1}{2}ram_c)$, $\Delta_\mu^{(n)}$ is the lattice discretization of the n'th gauge covariant derivative D_μ^n, and $F_{\mu\nu}$ is an a^2-improved clover discretization of the gluon field strength. All operators are tadpole improved; at tree level the couplings $c_F = c_3 = c_4 = 1$.

We have tested this action on lattices with $a = 0.25\,\mathrm{fm}$ and $0.4\,\mathrm{fm}$ [14]. We limited our study to quark masses near the strange quark mass. Thus we avoid the inessential complications of chiral extrapolation. Hadrons made of quarks with larger masses are generally smaller and heavier; consequently discretization errors are probably larger for the strange quark than for the other light quarks.

The leading correction in $M_{\rm D234c}$ beyond those in the SW formalism is

$$-\,c_3\,\frac{a^2}{6}\sum_\mu \Delta_\mu^{(3)}\gamma_\mu. \qquad (28)$$

This violates Lorentz invariance; it cancels similar violations in the leading-order terms. Thus we tested this correction by computing

$$c^2(\mathbf{p}) \equiv \frac{E_h^2(\mathbf{p}) - E_h^2(0)}{\mathbf{p}^2} \qquad (29)$$

where $E_h(\mathbf{p})$ is the energy of hadron h at three-momentum $\mathbf{p}$. In a Lorentz invariant theory, $c(\mathbf{p})$ should equal 1, the speed of light, for all $\mathbf{p}$. Results at $0.4\,\mathrm{fm}$ are shown in Fig. 5 for both the D234c and SW actions. D234c is dramatically superior, deviating from 1 by only 3–4% at $\mathbf{p}=0$. Also the dispersion relation is accurate to within 10% out to three-momenta of order $1.5/a$.

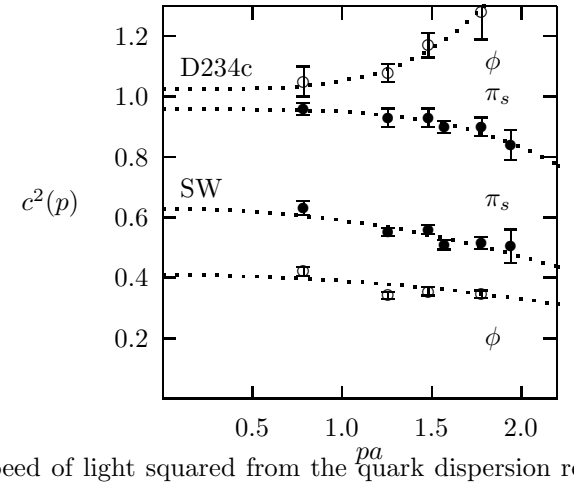

Figure 5: Speed of light squared from the quark dispersion relation using the D234c and SW actions, for pseudoscalar ("π_s") and vector ("ϕ") mesons made of strange quarks. Simulations are on isotropic lattices with $a \approx .4\,\mathrm{fm}$ and an improved gluon action. Fits shown are to $c_0 + c_4(pa)^4$ for D234c and $c_0 + c_2(pa)^2$ for SW.

We could tune the coupling c_3 in the D234c action nonperturbatively until $c^2(0) = 1$, thereby restoring Lorentz symmetry through order a^2. In the tadpole-improved theory at 0.4 fm, however, $c_3 = 1$ is so close to the correct value there is no point. There is no value of c_3 for which both the pseudoscalar and vector have exactly $c^2 = 1$; uncorrected a^4 errors are larger than the radiative corrections to c_3 at this lattice spacing.

There are two different definitions of a hadron's mass in lattice simulations. One is the "static mass," $E_h(0)$, and the other the "kinetic mass," which equals $E_h(0)/c^2(0)$. In D234c these two definitions agree to within 3-4%. In SW they differ by 40–60%. The kinetic and static masses in either formalism must be equal for zero mass mesons, because of the axis-interchange symmetry of the actions. The deviations seen here are because the strange quark is relatively massive at 0.4 fm; the ϕ mass, for example, is $2.1/a$ with D234c.

Another test of the D234c action is to vary the parameter r. The r-dependent terms in the D234c action are approximately redundant at low momenta, since they originate from a field transformation. We ran simulations both at the standard value $r = 1$ and at $r = 2/3$ to check this. With D234c the hadron masses changed by no more than 1%. Reducing r in this way lowers one of the ghost branches of the quark dispersion relation from an energy near $2/a$ to one near $1/a$. The lowering of the ghost changed little in the spectrum; our results are not strongly affected by ghost artefacts.

With SW, where the r terms are less highly corrected and therefore less

redundant, we found shifts of order 1–5% when we varied r as above.

The coefficients c_F, c_3, and c_4 are all renormalized by residual quantum effects, even after tadpole improvement. We varied these couplings to assess the importance of the potential corrections. Only c_F had a strong effect on the static hadron masses. D234c with $c_F = 1$ gives scaling errors in the ϕ mass of order 5% at 0.25 fm and 15% at 0.4 fm; but a radiative correction of $0.5\alpha_V$ in c_F, like that in the SW action (Eq. (14)), would shift these masses by 4–8%. Thus to achieve precisions of order a few percent at such large lattice spacings, we require a calculation of the order α_V correction to $c_{\rm sw}$. This calculation will be completed in the very near future.

The highly corrected D234c action is far superior to SW for states with high energies, either because $\mathbf{p}$ is large or because a is large. This makes it potentially useful for simulations with very large lattice spacings, large quark masses, or high-momentum states (for example, high-energy weak interaction form factors). Anisotropic versions of D234c have also been tested [14, 1].

4.5 Improved Staggered Quarks

The standard staggered-quark action, like the SW action, has leading errors of order a^2. Some attempt has been made to correct the a^2 errors [18], but this effort included only the obvious correction to the derivative in the action. Surprisingly, there are, in addition, four-quark interactions that must be included at tree-level in order a^2. Tree-level improvement of a lattice action requires that we match on-shell, tree-level, lattice scattering amplitudes with continuum QCD at low external momentum. Normally this implies that the lattice theory has no tree-level four-quark interactions since continuum QCD has none: in a tree-level quark-quark scattering amplitude, low-momentum external quarks imply a low-momentum internal gluon, and therefore these amplitudes will match automatically if the quark-gluon vertices match. This reasoning is incorrect for staggered quark formalisms. For example, a pair of low-momentum quarks can scatter exchanging a *high-energy* gluon with momentum $q = (\pi/a, 0, 0, 0)$, creating a pair of low-momentum quarks of a different flavor. Such flavor changing quark-quark amplitudes are absent in QCD and so must be eliminated by adding correction terms to the action. Since the gluon is highly virtual in a flavor-changing process, the correction term is a four-quark operator with a coefficient of a constant times $a^2\alpha_V(q_g)$, where the constant is computed from the scattering amplitude and q_g is the gluon momentum.

The four-quark corrections obviously will greatly reduce the flavor-symmetry violations observed in hadronic spectra from staggered quarks, since these are proportional to a^2. Supporting evidence also comes from the numerical experiments described in [18]. In these experiments, the link operator in the standard staggered-quark action was replaced by a smeared link. Smearing the links inhibits the exchange of high-momentum gluons by lowering the effective ultraviolet cutoff of the gluons; but the flavor-changing interaction requires ex-

changed gluons with the highest momenta on the lattice, $\pm\pi/a$ in one or more directions. Thus smearing should dramatically reduce the flavor splittings; and it does [18].

If the four-quark operators are computed and can be included in a practical action, the staggered-quark formalism will be more accurate than the SW action. Adding the four-quark corrections may also significantly reduce the size of α_V corrections to the mass renormalization for staggered quarks. This correction term has normal size for $r \approx 1$, but becomes quite large in the staggered-quark limit, $r = 0$. Thus the large size in this limit seems to be related to flavor mixing. A reduction of this renormalization is important for the determination of the strange-quark mass.

5 Conclusions

Techniques for simulating on coarser lattices are rapidly gaining acceptance within the lattice QCD community, and work at lattice spacings in the range of 0.1–0.2 fm is becoming widespread, again. It is likely that some combination of perturbatively improved actions, nonperturbative improvement, perfect actions, anisotropic lattices, Monte Carlo Renormalization Group, and the like will allow us to simulate accurately and efficiently on still coarser lattices in the very near future.

References

[1] For a review see G.P. Lepage, hep-lat/9607076, in the proceedings of the 1996 Schladming School on *Perturbative and Nonperturbative Aspects of Quantum Field Theory*.

[2] G.P. Lepage and P.B. Mackenzie,*Phys. Rev.* D **48**, 2250 (1993).

[3] See J. Sloan's lecture at this meeting, and S. Collins, R.G. Edwards, U.M. Heller, J. Sloan, *Nucl. Phys.* B(Proc. Suppl.) **53**, 877 (1997); hep-lat/9611022; and private communication.

[4] G.P. Lepage, *Nucl. Phys.* B(Proc. Suppl.) **47**, 3 (1996).

[5] See R. Sommer's lecture at this meeting, and M. Luscher, S. Sint, R. Sommer, P. Weisz, U. Wolff, *Nucl. Phys.* B **491**, 323 (1997), and references therein.

[6] C.T.H. Davies, K. Hornbostel, G.P. Lepage, P. McCallum, J. Shigemitsu, and J. Sloan, hep-lat/9703010, to be published in Phys. Rev. D.

[7] Colin Morningstar has verified the first-order coefficient and computed the q^* (private communication).

[8] P. Weisz, *Nucl. Phys.* B(Proc. Suppl.) **47**, 71 (1996).

[9] W.E. Caswell, G.P. Lepage, and J.R. Sapirstein, *Phys. Rev. Lett.* **38**, 488 (1978).

[10] See J. Shigemitsu's talk at this meeting, for example.

[11] W. Dimm, Cornell Ph.D. Thesis (1995); W. Dimm, G.P. Lepage, and P.B. Mackenzie, *Nucl. Phys.* B(Proc. Suppl.) **42**, 403 (1995); H. Trottier and G.P. Lepage, in preparation.

[12] P. Lepage private communication. This work was on anisotropic lattices with $a_s/a_t = 0.5$ and $a_s \approx .4\,\mathrm{fm}$.

[13] H. Trottier, *Phys. Rev.* D **55**, 6844 (1997).

[14] M. Alford, T. Klassen and G.P. Lepage, in preparation. See also, by the same authors, *Nucl. Phys.* B(Proc. Suppl.) **47**, 370 (1996); *Nucl. Phys.* B(Proc. Suppl.) **53**, 861 (1997); hep-lat/9611010, to be published in Phys. Rev. D.

[15] M. Alford, T. Klassen, G.P. Lepage, C. Morningstar and M. Peardon, in preparation.

[16] C. Morningstar and M. Peardon, hep-lat/9704011.

[17] N. Shakespeare and H. Trottier, in preparation.

[18] See D. Toussaint's lecture at this meeting.